SOLUTION to February 2021 exam
Financial Econometrics A

Question A:

Consider the ARCH model given by

$$y_t = \delta_t(\beta, \gamma)^{1/2} x_t,\ x_t = \sigma_t(\alpha) z_t,$$
$$\sigma_t^2(\alpha) = 1 - \alpha + \alpha x_{t-1}^2,$$
$$\delta_t(\beta, \gamma) = \beta + \gamma d(t),\ d(t) = \frac{t^2}{1+t^2},$$

with z_t iid $N(0,1)$ distributed, x_0 fixed and $t = 1, 2, ..., T$. Also $0 \leq \alpha < 1$, $\beta > 0$ and $\gamma \geq 0$.

As to the role of $\delta_t(\beta, \gamma)$, observe that $d(t) \in (0,1)$ and hence $\delta_t(\beta, \gamma) \in (\beta, \beta + \gamma)$.

Question A.1: Show that (using as usual the notation that δ_t and σ_t^2 denote $\delta_t(\beta, \gamma)$ and $\sigma_t^2(\alpha)$ respectively evaluated at the true values α_0, β_0 and γ_0),

$$E(y_t|x_{t-1}) = 0 \text{ and } V(y_t|x_{t-1}) = \delta_t \sigma_t^2$$

Show furthermore that

$$V(y_t) = \beta_0 + \gamma_0 d(t),$$

and state a sufficient condition on α_0 for this to hold.

Discuss the role of $\delta_t(\beta, \gamma)$. In particular discuss what happens if $\gamma_0 = 0$ and $\gamma_0 > 0$ respectively.

Solution Question A.1: By definition

$$E(y_t|x_{t-1}) = \delta^{1/2}(t)\sigma_t E(z_t) = 0$$
$$V(y_t|x_{t-1}) = \delta(t)\sigma_t^2 E(z_t^2) = \delta(t)\sigma_t^2$$

and as $V(x_t) = \frac{1-\alpha_0}{1-\alpha_0} = 1$ if $\alpha_0 < 1$, it follows that

$$V(y_t) = \delta(t) V(x_t) = \beta_0 + \gamma_0 d(t).$$

When $\gamma_0 = 0$, this is a simple ARCH(1) model reparametrized such that $V(y_t) = \beta_0$, while if $\gamma_0 > 0$ the variance is time varying and increasing for the sample. This may be of interest when modeling volatility with an underlying deterministic and bounded trend.

Question A.2: Fix all parameters at their true values, except γ and α. Show that the first order derivatives of the log-likelihood function is given by,

$$S_T^\gamma = \partial L(\alpha,\gamma)/\partial\gamma|_{\alpha=\alpha_0,\gamma=\gamma_0} = -\frac{1}{2}\sum_{t=1}^T s_t^\gamma, \quad s_t^\gamma = \frac{d(t)}{\delta_t}\left(1-z_t^2\right).$$

$$S_T^\alpha = \partial L(\alpha,\gamma)/\partial\alpha|_{\alpha=\alpha_0,\gamma=\gamma_0} = -\frac{1}{2}\sum_{t=1}^T s_t^\alpha, \quad s_t^\alpha = \frac{y_{t-1}^2/\delta_t}{1-\alpha_0+\alpha_0 y_{t-1}^2/\delta_t}\left(1-z_t^2\right).$$

Argue that $E(s_t^\gamma) = E(s_t^\alpha) = 0$.

Solution Question A.2:
The derivative follows by the likelihood function is given by

$$L(\gamma) = -\frac{1}{2}\sum_{t=1}^T\left(\log\sigma_t^2 + \log\delta_t(\beta_0,\gamma) + \frac{y_t^2}{\sigma_t^2\delta_t(\beta_0,\gamma)}\right),$$

and using $y_t = \delta_t^{1/2}\sigma_t z_t$ repeatedly.

Question A.3: Show that for $\alpha_0 \in (0,1)$,

$$T^{-1/2}\sum_{t=1}^T s_t^\alpha \xrightarrow{D} N(0,\xi\kappa) \quad \kappa = E\left(1-z_t^2\right)^2 = 3 \text{ and } \xi = E\left(\frac{y_{t-1}^2/\delta_t}{1-\alpha_0+\alpha_0 y_{t-1}^2/\delta_t}\right)^2.$$

Explain why $\xi < \infty$.

Solution Question A.3:
By Question A.2: $s_t^\alpha = \frac{y_{t-1}^2/\delta_t}{1-\alpha_0+\alpha_0 y_{t-1}^2/\delta_t}(1-z_t^2) = \frac{x_{t-1}^2}{1-\alpha_0+\alpha_0 x_{t-1}^2}(1-z_t^2)$ and hence routine arguments (provide arguments) give the desired as x_t is weakly mixing and $\frac{x_{t-1}^2}{1-\alpha_0+\alpha_0 x_{t-1}^2} < 1/\alpha_0$. This implies the desired for ξ.

Question A.4: It can be shown that for $0 < \alpha_0 < 1$ and $\gamma_0 > 0$,

$$T^{1/2}\sum_{t=1}^T(\hat{\alpha}-\alpha_0, \hat{\gamma}-\gamma_0)$$

is asymptotically Gaussian. What would you expect the limiting distribution of the likelihood ratio statistic for $H : (\gamma = \gamma_0, \ \ \alpha = \alpha_0)$ is?

Solution Question A.4:

χ_2^2 type if α_0,$\gamma_0 > 0$. Otherwise, if (be precise) $\alpha_0 = 0$ and/or $\gamma_0 = 0$ would expect $\frac{1}{2}\chi_2^2$, or $\frac{1}{2}\chi_1^2$ - or similar.

Question B:

Consider the model for $y_t \in \mathbb{R}$ given by

$$y_t = 1_{(s_t=1)} z_{t,1} + 1_{(s_t=2)} z_{t,2},$$

where

$$1_{(s_t=i)} = \begin{cases} 1 & \text{if } s_t = i \\ 0 & \text{if } s_t \neq i \end{cases}, \quad i = 1, 2,$$

and, for $i = 1, 2$, $(z_{t,i})$ is an i.i.d. process with

$$z_{t,i} \stackrel{d}{=} t_{v_i}, \quad i = 1, 2,$$

i.e. $z_{t,i}$ is Student's t-distributed with $v_i > 2$ degrees of freedom. The processes $(z_{t,1})$ and $(z_{t,2})$ are independent. Moreover, (s_t) is a two-state Markov chain with transition probabilities

$$P(s_t = j | s_{t-1} = i) = p_{ij} \in [0, 1], \quad i, j = 1, 2.$$

Assume throughout that (s_t) is independent of the processes $(z_{t,1})$ and $(z_{t,2})$.

Lastly, recall that if X is Student's t-distributed with $v > 0$ degrees of freedom, then the density of X is

$$f(x) = \frac{\Gamma\left(\frac{v+1}{2}\right)}{\Gamma\left(\frac{v}{2}\right)\sqrt{\pi v}} \left(1 + \frac{x^2}{v}\right)^{-\frac{v+1}{2}},$$

where $\Gamma(\cdot)$ is the gamma function.

Question B.1: Give a brief interpretation of the model.
Argue that (y_t, s_t) is a Markov chain.
Argue that the conditional density of (y_t, s_t) satisfies

$$f((y_t, s_t) | y_{t-1}, s_{t-1}) = f(y_t | s_t) f(s_t | s_{t-1}).$$

Solution: Interpretation: Markov-switching model where y_t "switches" between two t-distributions, e.g. the variance and tail-heaviness of y_t is

state-dependent.
Moreover,

$$\begin{aligned} f((y_t, s_t)|y_{t-1}, s_{t-1}, y_{t-2}, s_{t-2}, \ldots) &= f(y_t|s_t, y_{t-1}, s_{t-1}, y_{t-2}, s_{t-2}, \ldots) \\ &\quad \times f(s_t|y_{t-1}, s_{t-1}, y_{t-2}, s_{t-2}, \ldots) \\ &= f(y_t|s_t) \times f(s_t|s_{t-1}) \\ &= f((y_t, s_t)|s_{t-1}) \\ &= f((y_t, s_t)|s_{t-1}, y_{t-1}), \end{aligned}$$

where the second equality follows by the model structure (i.e. that the processes $(z_{t,i})$ are i.i.d. and independent of (s_t), and that (s_t) is a Markov chain), and the last equality follows by the fact that (y_t, s_t) and y_{t-1} are independent conditional on s_{t-1}.

Question B.2: Suppose that the Markov chain (s_t) is irreducible and aperiodic. Show that

$$E[y_t] = 0$$

and

$$E[y_t^2] = \frac{1-p_{22}}{2-p_{11}-p_{22}} \left(\frac{v_1}{v_1 - 2} \right) + \frac{1-p_{11}}{2-p_{11}-p_{22}} \left(\frac{v_2}{v_2 - 2} \right).$$

Solution: It holds that

$$E[y_t] = \sum_{i=1}^{2} P(s_t = i) \underbrace{E[z_{t,i}]}_{=0}$$

and

$$E[y_t^2] = \sum_{i=1}^{2} P(s_t = i) E[z_{t,i}^2] = \sum_{i=1}^{2} P(s_t = i) \frac{v_i}{v_i - 2},$$

and using that (s_t) is irreducible and aperiodic, we have that

$$P(s_t = 1) = \frac{1-p_{22}}{2-p_{11}-p_{22}} = 1 - P(s_t = 2).$$

Question B.3: Suppose that we want to estimate the model parameters $\theta = (v_1, v_2)'$. Based on a sample $(y_0, y_1, \ldots, y_T)$ the log-likelihood function (conditional on y_0) is given by

$$L_T(\theta) = \sum_{t=1}^{T} \log f(y_t|y_{t-1}, \ldots, y_0).$$

Show that

$$f(y_t|y_{t-1},\ldots,y_0)=\sum_{i=1}^{2}f(y_t|s_t=i)P(s_t=i|y_{t-1},\ldots,y_0),$$

with

$$f(y_t|s_t=i)=\frac{\Gamma\left(\frac{v_i+1}{2}\right)}{\Gamma\left(\frac{v_i}{2}\right)\sqrt{\pi v_i}}\left(1+\frac{y_t^2}{v_i}\right)^{-\frac{v_i+1}{2}}.$$

Explain briefly how you would compute $P(s_t=i|y_{t-1},\ldots,y_0)$.

Solution: The expression for $f(y_t|y_{t-1},\ldots,y_0)$ follows by straightforward derivations. The density $f(y_t|s_t=i)$ follows by noting that y_t is Student's t distributed with v_i degrees of freedom conditional on $\{s_t=i\}$. The predicted probability $P(s_t=i|y_{t-1},\ldots,y_0)$ may be computed using a filtering algorithm. Some details should be provide, e.g. a brief outline of the algorithm.

Question B.4: Let $\tau_{\text{risk}}>0$ denote some constant risk threshold, and define the (conditional) probability of a loss exceeding τ_{risk} at time $T+1$,

$$\varsigma_{T+1}(\tau_{\text{risk}})\equiv P(-y_{T+1}\geq\tau_{\text{risk}}|y_T,y_{T-1},\ldots,y_0).$$

Let $\mathcal{T}_i:\mathbb{R}\to[0,1]$ denote the cdf of a Student's t-distribution with v_i degrees of freedom, $i=1,2$.

Show that

$$\varsigma_{T+1}(\tau_{\text{risk}})=\sum_{i=1}^{2}\mathcal{T}_i(-\tau_{\text{risk}})P(s_{T+1}=i|y_T,y_{T-1},\ldots,y_0).$$

Discuss briefly how you would estimate $\varsigma_{T+1}(\tau_{\text{risk}})$ based on a sample $(y_0,\ldots,y_T)$.

Solution: We have that

$$\begin{aligned}
&P(-y_{T+1}\geq\tau_{\text{risk}}|y_T,y_{T-1},\ldots,y_0)\\
&\quad=\sum_{i=1}^{2}P(-y_{T+1}\geq\tau_{\text{risk}}|s_{T+1}=i)P(s_{T+1}=i|y_T,y_{T-1},\ldots,y_0)\\
&\quad=\sum_{i=1}^{2}P(z_{T+1,i}\leq-\tau_{\text{risk}}|s_{T+1}=i)P(s_{T+1}=i|y_T,y_{T-1},\ldots,y_0)\\
&\quad=\sum_{i=1}^{2}\mathcal{T}_i(-\tau_{\text{risk}})P(s_{T+1}=i|y_T,y_{T-1},\ldots,y_0).
\end{aligned}$$

The predicted probabilities $P(s_{T+1} = i | y_T, y_{T-1}, \ldots, y_0)$ may be computed using the filtering algorithm mentioned in the previous question. The probabilies $\mathcal{T}_i(-\tau_{\text{risk}})$ may be computed using the estimates of the degrees of freedom v_i.